are not shown here for lack of space‡). For the discontinuous branch, $F'(\eta)$ exhibits both reverse flow and velocity overshoot when $C=-0.75$ and $f_w=-0.5$ for all values of ω and g_w; there is no reverse flow when $C=-0.5$. Similar behavior has been observed by Libby[1] for $\omega=1$. For the continuous branch, the reverse flow occurs for both $C=-0.75$ and -0.5, but there is no velocity overshoot, for any value of C, ω, f_w, and g_w. The effect of decreasing ω from 1 to 0.5 is to decrease the reverse flow for both branches and to increase the velocity overshoot in the discontinuous branch (there is no velocity overshoot in the continuous branch). The effect is more pronounced for $g_w=0.2$ than for $g_w=0.6$. The occurrence of reverse flow and velocity overshoot is due to the combined effects of inertia, pressure, and shear in the boundary layer.[1] For both branches of solutions, the effect of increasing g_w is to increase the magnitude of reverse flow, whereas the effect of increasing f_w ($f_w<0$) or $C(C<0)$ is to decrease it.

The skin-friction and heat-transfer parameters $f_l''(0)$, $F_l''(0)$, and $G'(0)$ for discontinuous branch for various values of the parameters are given in Table 1. The corresponding results for the continuous branch have also been obtained, but they are not given here for lack of space.‡ It is observed that $f_l''(0)$, $F_l''(0)$, and $G'(0)$ increase as ω decreases whatever may be the values of C, f_w, and g_w, but the effect of ω on them is more pronounced for lower values of g_w. For a given C, $f_l''(0)$, $F_l''(0)$, and $G'(0)$ decrease as f_w ($f_w<0$) decreases. It is also seen that $f_l''(0)$ and $G'(0)$ increases but $F_l''(0)$ decreases as C decreases and $F_l''(0)<0$ for some critical value of C depending on the magnitude of ω, f_w, and g_w. It may be noted that in the continuous branch the behavior of $f_l''(0)$ and $G'(0)$ for some values of f_w and g_w is opposite to that of the discontinuous branch until a critical value of C is reached for which $F_l''(0)=0$. Beyond this critical value of C, similar to the discontinuous case, $F_l''(0)$ and $G'(0)$ increase as C decreases. It is further observed that $f_l''(0)$ increases while $F_l''(0)$ decreases as g_w increases for all values of $\omega, C(C\leq-0.5)$, f_w ($f_w<0$), but $G'(0)$ increases or decreases as g_w increases depending on the values of C and f_w. We have compared our results for the continuous branch with those tabulated by Wortman et al.[2] and Vimala and Nath[3] and for the discontinuous branch (for $\omega=1$ and $g_w=0$§) with those tabulated by Libby[1] and found them to be in good agreement.

Conclusions

The skin friction both in the principal and transverse directions and heat transfer are significantly affected by the variation of the density-viscosity product ($\omega\neq1$) across the boundary layer at low-wall temperatures. Furthermore, the skin friction in the transverse direction is strongly dependent on the nature of the stagnation point. The velocity profiles in the transverse direction exhibit both reverse flow and velocity overshoot, and the effect of the variation of the density-viscosity product is to decrease the reverse flow and to increase the velocity overshoot.

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Improved Algebraic Relation for the Calculation of Reynolds Stresses

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I. Introduction

THE prediction capability of a turbulence model depends on how effectively one can prescribe the Reynolds stress distribution in closing the system of equations. The simplest and most widely used has been the Boussinesq treatment of the Reynolds stresses. As is now well-known, the Boussinesq hypothesis holds only when the strain rates are fairly small. The main reason is that by construction the Boussinesq formula implies that the principal axes of the Reynolds stress tensor are parallel to the principal axes of the strain rate tensor, so that any change in the strain rate is directly felt in the stresses. This instantaneous change of the Reynolds stresses with the strain rates is not supported by the experimental observations, because the Reynolds stresses, being due to the vorticity fluctuations, require some time to adjust to the new strain rates. In order to overcome these shortcomings, one must either abondon the Boussinesq hypothesis altogether and solve the six Reynolds transport equations, which is costly in terms of computer time, or improve upon the hypothesis itself.

In this paper, we follow a recent analysis of Rodi[1] to construct an improved second-order version of the Boussinesq hypothesis. An algebraic relation for the turbulent stresses has been obtained through a consideration of the transport equations of the Reynolds stresses. Consequently, the resulting relation has the necessary influence of the convective and diffusive transport effects of a turbulence stress field.

II. Analysis

The transport equations of the Reynolds stresses ($\overline{-u_i u_j}$) and the equation of turbulence energy ($\bar{e}=\frac{1}{2}\overline{u_i u_i}$) for an incompressible flow, respectively, are

$$d\tau_{ij}/dt=P_{ij}+Q_{ij}+D_{ij}-\epsilon_{ij}, \quad \tau_{ij}=\overline{u_i u_j} \tag{1}$$

$$d\bar{e}/dt=P+D-\epsilon \tag{2}$$

where d/dt is the substantive derivative based on the mean velocity components U_i; P_{ij}, D_{ij}, ϵ_{ij}, respectively, are the production, diffusion, and dissipation of the Reynolds stresses, Q_{ij} is the pressure-strain correlation, and P, D, and ϵ, respectively, are the production, diffusion, and dissipation of the turbulence energy. In this paper, we have used the modeling of the terms Q_{ij}, D_{ij}, ϵ_{ij}, and D as reported in Refs. 1 and 2, which, when using the summation convention on

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repeated indices, are

$$Q_{ij} = \frac{c_I \epsilon}{\bar{e}} (\tfrac{2}{3}\bar{e}\delta_{ij} - \tau_{ij}) + \gamma (\frac{2P}{3} \delta_{ij} - P_{ij}) \tag{3}$$

$$D_{ij} = \frac{\partial}{\partial x_k} (\nu \frac{\partial \tau_{ij}}{\partial x_k} + \frac{c_s \bar{e}}{\epsilon} \tau_{k\ell} \frac{\partial \tau_{ij}}{\partial x_\ell}) \tag{4}$$

$$\epsilon_{ij} = \tfrac{2}{3}\epsilon \, \delta_{ij} \tag{5}$$

$$D = \frac{\partial}{\partial x_k} (\nu \frac{\partial \bar{e}}{\partial x_k} + \frac{c_s \bar{e}}{\epsilon} \tau_{k\ell} \frac{\partial \bar{e}}{\partial x_\ell}) \tag{6}$$

where c_I, γ, and c_s are empirical constants, and ν is the kinematic viscosity. The terms P_{ij}, P, and ϵ are

$$P_{ij} = -[\tau_{ik} (\partial U_j / \partial x_k) + \tau_{jk} (\partial U_i / \partial x_k)] \tag{7}$$

$$P = \tfrac{1}{2} P_{ii} = -\tau_{k\ell} (\partial U_k / \partial x_\ell) \tag{8}$$

$$\epsilon = \nu (\overline{\partial u_I / \partial x_j})^2 \tag{9}$$

Introducing the notation

$$T_{ij} = \tau_{ij} / \bar{e} \tag{10}$$

and arranging terms in D_{ij}, we have

$$D_{ij} = \nu \frac{\partial}{\partial x_k} (\bar{e} \frac{\partial T_{ij}}{\partial x_k}) + \nu \frac{\partial \bar{e}}{\partial x_k} \frac{\partial T_{ij}}{\partial x_k} + D T_{ij}$$
$$+ \frac{\partial}{\partial x_k} \left\{ \frac{c_s (\bar{e})^2}{\epsilon} \tau_{k\ell} \frac{\partial T_{ij}}{\partial x_\ell} \right\} + \frac{c_s \bar{e}}{\epsilon} \tau_{k\ell} \frac{\partial \bar{e}}{\partial x_\ell} \frac{\partial T_{ij}}{\partial x_k} \tag{11}$$

Introducing Eq. (11), the identity

$$\frac{d\tau_{ij}}{dt} = \bar{e} \frac{dT_{ij}}{dt} + T_{ij} \frac{d\bar{e}}{dt}$$

and Eq. (2) in Eq. (1), and neglecting the derivatives of T_{ij} in comparison with the other terms, we obtain

$$T_{ij} (P - \epsilon) = P_{ij} + Q_{ij} - \epsilon_{ij} \tag{12}$$

Upon substituting Eq. (3) and (5) in Eq. (12), we obtain

$$T_{ij} = \tfrac{2}{3}\delta_{ij} + \gamma_0 (P_{ij}/\epsilon - \tfrac{2}{3}\delta_{ij} P/\epsilon) / (d_I + P/\epsilon) \tag{13}$$

where

$$\gamma_0 = 1 - \gamma \quad \text{and} \quad d_I = c_I - 1$$

We now introduce the following notation

$$\theta^2 = \tfrac{1}{2} \overline{\omega_i \omega_i} \qquad M_{ij} = (1/\theta) (\partial U_i / \partial x_j) \tag{14}$$

where θ is the vorticity density and ω_i is the fluctuating vorticity component. It follows directly from the Kolmogorov-Saffman equation of energy[3] that the dissipation of energy ϵ is given by

$$\epsilon = \bar{e}\theta \tag{15}$$

Using Eqs. (14) and (15) in Eqs. (7) and (8), we get

$$P_{ij}/\epsilon = -(T_{ik} M_{jk} + T_{jk} M_{ik}) \tag{16}$$

$$P/\epsilon = -T_{k\ell} M_{k\ell} \tag{17}$$

If Eqs. (16) and (17) are substituted in Eq. (13), then we get a system of nonlinear simultaneous algebraic equations for the determination of T_{ij}. Rodi,[1] in his derivation, did not use the expansion (17), but retained P/ϵ as a paramter, and solved Eq. (13) for T_{ij}. Since P/ϵ contains all T_{ij}'s, we follow an approach different from that of Rodi, which, in the first place, establishes the validity of the Boussinesq hypothesis, and in the second place yields an improved algebraic relation for T_{ij}.

Equation (13) can be written in various iterative forms; however, the following form is chosen because it yields various approximations in a direct fashion

$$d_I \, T_{ij}^{(n+1)} - T_{ij}^{(n)} T_{k\ell}^{(n)} M_{k\ell}$$
$$= \tfrac{2}{3}\delta_{ij} (d_I - T_{k\ell}^{(n)} M_{k\ell}) + \gamma_0 (-T_{ik}^{(n)} M_{jk} - T_{jk}^{(n)} M_{ik}$$
$$+ \tfrac{2}{3}\delta_{ij} T_{k\ell}^{(n)} M_{k\ell}) \tag{18}$$

where n is the iteration index.

For the zeroeth approximation, we take

$$T_{ij}^{(0)} = \tfrac{2}{3}\delta_{ij}$$

and by using the continuity equation $M_{ii} = 0$, we get

$$T_{ij}^{(1)} = \tfrac{2}{3}\delta_{ij} - \alpha_0^2 (M_{ij} + M_{ji}) \tag{19}$$

where

$$\alpha_0^2 = 2\gamma_0 / 3 d_I = \text{const} \tag{20}$$

Equation (19) is the usual Boussinesq formulation, which also has been used, among others, by Kolmogorov[4] and Saffman.[3] It is only a first approximation and is expected to hold in situations in which the mean flow is not changing rapidly.

In order to obtain the second approximation, we introduce Eq. (19) in Eq. (18) and neglect terms of the third-order in M_{ij}

$$T_{ij}^{(2)} = \tfrac{2}{3}\delta_{ij} - \alpha_0^2 (M_{ij} + M_{ji})$$
$$+ 3/2\alpha_0^4 [(M_{ik} + M_{ki}) M_{jk} + (M_{jk} + M_{kj}) M_{ik}$$
$$- \tfrac{2}{3}\delta_{ij} (M_{k\ell} + M_{\ell k}) M_{k\ell}] \tag{21}$$

Equation (21) provides the second approximation to the Reynolds stresses and is expected to hold from low to moderate variations of the strain rates.

By following a philosophically different approach, Saffman[3] also has obtained an expression similar to Eq. (21), which, in our notation, is

$$T_{ij}^{(2)} = \tfrac{2}{3}\delta_{ij} - \alpha^2 (M_{ij} + M_{ji})$$
$$+ (\lambda/2) [(M_{ik} + M_{ki}) M_{jk} + (M_{jk} + M_{kj}) M_{ik}$$
$$- \{ (M_{jk} + M_{kj}) M_{ki} + (M_{ki} + M_{ik}) M_{kj} \}] \tag{22}$$

where α and λ are constants. Comparing Eqs. (21) and (22), we find that although both are in dimensional agreement, they differ in their last terms. It must be noted that Eq. (21) is a consequence of the complete Navier-Stokes equations, whereas Eq. (22) is an attempt at finding a relaxation model to overcome the deficiencies of the first approximation.

A comparison of Eqs. (21) and (22) yields the values of the constant α_0 appearing in (21). Thus we have

$$\alpha_0 = \alpha = 0.3$$

The value of $\alpha_0 = 0.3$ has been used by Saffman,[3] and also by Pope and Whitelaw,[5] but from Eq. (20) we find that the value $\alpha_0 = 0.3$ is not consistent with the values $\gamma_0 = 0.4$ and $d_I = 0.5$ as proposed in Ref. 2. However, if we take the value $d_I = 1.86 \simeq [(2/.7 - 1)$ as mentioned by Rotta,[6], and $\gamma_0 = 0.3$, then $\alpha_0 = 0.33$, which is near to the value used by Saffman. For the sake of clarity, we therefore select the value $\alpha_0 = 0.3$ in Eq. (21).

For two-dimensional mean flow, Eq. (21) yields the following expressions for the normal and shear stresses

$$T_{11}^{(2)} = T_{11}^{(1)} + \alpha_0^4 [2M_{11}^2 + 3(M_{12}+M_{21})M_{12} - (M_{12}+M_{21})^2] \quad (23a)$$

$$T_{22}^{(2)} = T_{22}^{(1)} + \alpha_0^4 [2M_{11}^2 + 3(M_{12}+M_{21})M_{21} - (M_{12}+M_{21})^2] \quad (23b)$$

$$T_{33}^{(2)} = T_{33}^{(0)} - \alpha_0^4 [4M_{11}^2 + (M_{12}+M_{21})^2] \quad (23c)$$

$$T_{12}^{(2)} = T_{12}^{(1)} - 3\alpha_0^4 M_{11}(M_{12}-M_{21}) \quad (23d)$$

On the other hand, Saffman's equation (22) yields

$$T_{11}^{(2)} = T_{11}^{(1)} + \lambda(M_{12}+M_{21})(M_{12}-M_{21}) \quad (24a)$$

$$T_{22}^{(2)} = T_{22}^{(1)} - \lambda(M_{12}+M_{21})(M_{12}-M_{21}) \quad (24b)$$

$$T_{33}^{(2)} = T_{33}^{(0)} \quad (24c)$$

$$T_{12}^{(2)} = T_{12}^{(1)} - 2\lambda M_{11}(M_{12}-M_{21}) \quad (24d)$$

where λ is an emperical constant. Thus, although the shear stresses [Eqs. (23d) and (24d)] have the same distributions, the normal stresses [Eqs. (23a-23c) and (24a-24c)] have entirely different distributions.

In the wall region $M_{11} = M_{22} = M_{21} \simeq 0$ and

$$M_{12} \simeq 1/\alpha_0$$

so that Eqs. (23) become

$$T_{11}^{(2)} - 2/3 = 2\alpha_0^2 \quad (25a)$$

$$T_{22}^{(2)} - 2/3 = -\alpha_0^2 \quad (25b)$$

$$T_{33}^{(2)} - 2/3 = -\alpha_0^2 \quad (25c)$$

$$T_{12}^{(2)} = -\alpha_0 \quad (25d)$$

Equations (24) become

$$T_{11}^{(2)} - 2/3 = \lambda/\alpha_0^2 \quad (26a)$$

$$T_{22}^{(2)} - 2/3 = -\lambda/\alpha_0^2 \quad (26b)$$

$$T_{33}^{(2)} = 2/3 \quad (26c)$$

$$T_{12}^{(2)} = -\alpha_0 \quad (26d)$$

Table 1 Comparison of the near-wall data

	Present ($\alpha = 0.3$)	Present ($\alpha = 0.34$)	Saffman[3]	Ref. 7
$T_{11}^{(2)} - 2/3$	0.18	0.23	.22	0.32
$T_{22}^{(2)} - 2/3$	−0.09	−0.12	−0.22	−0.18
$T_{33}^{(2)} - 2/3$	−0.09	−0.12	0.0	−0.10
$T_{12}^{(2)}$	−0.30	−0.34	−0.30	−0.34

Saffman[3] now takes $\lambda = 0.02$ to match the three normal stresses with the experimental data, which roughly are in the ratio 4:2:3. Numerical values based on Eqs. (25) and (26), and the experimental values as quoted in Ref. 7, are tabulated in Table 1.

A comparison of values in Table 1 shows that $\alpha_0 = 0.34$ in the present model may be more suitable than $\alpha_0 = 0.3$. However, any adjustment of the constant α_0 or the actual prediction capability of the proposed second-order algebraic relation (21) can be ascertained only after it has been used in the calculation of various turbulent flows.

Based on the works of Launder et al.[2] and on the most recent review by Reynolds,[8] it is possible to establish in advance the limitations of the second approximation, viz., Eq. (21). For example, it may be observed that in the wall region the normal stresses $T_{22}^{(2)}$ and $T_{33}^{(2)}$ are equal [Eqs. (25b) and (25c)]. This result is in exact conformity with Eq. (14) of Ref. 2 in which Launder et al. have shown that in any simpler pressure-strain hypothesis there is no direct production of T_{22} and T_{33}, and therefore they tend to be equal. Thus it can be stated roughly that the amount of approximation involved in Eq. (21) is the same as that involved in obtaining the Reynolds stresses through the six Reynolds stress transport equations, by using a simpler pressure-strain relation.

III. Conclusion

The purpose of this paper has been to demonstrate that the Boussinesq eddy viscosity hypothesis and its higher approximations are a direct consequence of the Reynolds stress transport equations. The basic assumption of the analysis is that the derivatives of the terms $T_{ij} = \tau_{ij}/\bar{e}$ are small in comparison with the other terms in the rate equation for τ_{ij}. Because of the implicit effects of the convection and diffusion in the second approximation, the simple algebraic relation (21) is expected to provide a basis for the prediction of complex flows.

It is important to mention that the validity of Eq. (13), which rests on the assumption that the rate of change of T_{ij} be small in comparison with the other terms, is not new either in the paper by Rodi[1] or in the present one. This idea§ has been used earlier by Donaldson,[9] who called it as the "super-equilibrium" limit. As an application, Donaldson obtained all of the Reynolds stress terms algebraically for a line vortex.

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